Explicit analytical solutions for one-dimensional steady state flow in layered, heterogeneous unsaturated soils under random boundary conditions

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Received 6 December 2005; revised 31 May 2007; accepted 13 June 2007; published 21 September 2007.

[1] In this study, we directly derive first-order analytical solutions to the pressure head moments (mean and variance) for one-dimensional steady state unsaturated flow in randomly heterogeneous layered soil columns under various random boundary conditions. We assume that the constitutive relation between the unsaturated hydraulic conductivity and the pressure head follows an exponential model, and treat the saturated hydraulic conductivity K_s as a random function and the pore size distribution parameter α as a random constant. Unlike the solution given in Lu and Zhang (2004) in which Kirchhoff transformation was used and the solution to pressure head variance was presented as a function of (cross-)covariances related to the intermediate, Kirchhoff-transformed variable, the solution to the pressure head variance presented in this paper is an explicit function of the input variabilities. In addition, we also give analytical solutions to the statistics of the unsaturated hydraulic conductivity and the effective water content. These first-order analytical solutions are compared with those from Monte Carlo simulations. We also investigated the effect of uncertain boundary conditions, the relative contribution of input variabilities to the head variance, and the possible errors introduced by treating the correlated α field as a random constant in the analytical solutions. The results indicate that the uncertain constant head at the bottom of a deep soil column may not have a significant effect on predicting flow statistics in the upper portion of the column. Furthermore, it is found that treating α as a random constant is justified when the correlation length of α is relatively large as compared to the layer thickness.

Citation: Lu, Z., D. Zhang, and B. A. Robinson (2007), Explicit analytical solutions for one-dimensional steady state flow in layered, heterogeneous unsaturated soils under random boundary conditions, *Water Resour. Res.*, *43*, W09413, doi:10.1029/2005WR004795.

1. Introduction

[2] Various analytical solutions for one-dimensional infiltration problems have been presented in the literature [e.g., *Warrick*, 1974; *Srivastava and Yeh*, 1991; *Tracy*, 1995; *Basha*, 1999]. In these solutions, it is assumed that soil properties either are homogeneous or vary deterministically in space. Quantification of uncertainties associated with unsaturated flow in randomly heterogeneous media is challenging. Most of the relevant studies are numerical, either by Monte Carlo simulations or numerical moment equation methods [*van Genuchten*, 1982; *Andersson and Shapiro*, 1983; *Yeh et al.*, 1985; *Hopmans et al.*, 1988; *Romano et al.*, 1998; *Zhang and Winter*, 1998; *Ferrante and Yeh*, 1999; *Foussereau et al.*, 2000; *Zhang*, 2002; *Lu et al.*, 2002; *Lu and Zhang*, 2002; *Zhang and Lu*, 2002]. Only a limited number of analytical solutions to the stochastic unsaturated flow problem are available in the literature. These solutions in general are restricted to single-layered, statistically homogeneous porous media. *Yeh et al.* [1985] used spectral representations of heterogeneous soil properties to derive the solutions of pressure head statistics for unsaturated flow in the gravity-dominated regime. *Zhang et al.* [1998] gave analytical solutions to the pressure head variance for gravity-dominated flow with both Gardner-Russo and Brooks-Corey constitutive models. *Indelman et al.* [1993] derived expressions for pressure head moments for one-dimensional steady state unsaturated flow in bounded single-layered heterogeneous formations under deterministic boundary conditions (a constant head at the bottom and constant flux at the top). These expressions contain integrals that have to be evaluated numerically in general.

[3] Because of the nonlinearity of unsaturated flow, the Kirchhoff transformation is often employed to linearize the equation of unsaturated flow. *Tartakovsky et al.* [1999], using the Kirchhoff transformation, solved the mean pressure head and the head variance for the one-dimensional unsaturated flow problem up to second order in terms of variability of the log saturated hydraulic conductivity. Although their equations are given in a more general form, the analytical solution for the one-dimensional problem is restricted to a special case of a single-layered soil column with

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0043-1397/07/2005WR004795$09.00

a deterministic pore size distribution parameter under deterministic boundary conditions. *Tartakovsky et al.* [2004] gave an analytical solution to the moments of the Kirchhoff-transformed variable for transient unsaturated flow in statistically homogeneous porous media with an assumption of a deterministic pore size distribution parameter. Since the analytical solutions given by *Tartakovsky et al.* [1999] and *Tartakovsky et al.* [2004] are under deterministic boundary conditions, these solutions are not applicable to multiple layered soil systems. Recently, using the Kirchhoff transformation, *Lu and Zhang* [2004] derived analytical solutions to the first two moments (mean and variance) of the pressure head for one-dimensional steady state unsaturated flow in layered, randomly heterogeneous soils. This was the first time in the literature that the analytical solutions of flow moments for the one-dimensional multiple layered unsaturated soil were given. Note that in both *Tartakovsky et al.* [2004] and *Lu and Zhang* [2004] the statistics of the pressure head are represented in terms of the statistics of the intermediate Kirchhoff-transformed variable rather than those of soil properties.

[4] In this paper, we try to solve the same problem as studied by *Lu and Zhang* [2004] directly without employing any transformations and give an analytical solution to the statistical moments of the pressure head for the multiple layered unsaturated heterogeneous soil column. We first present analytical solutions for the statistics (mean and variance) of the pressure head and the unsaturated hydraulic conductivity for one-dimensional steady state unsaturated flow in a single-layered heterogeneous soil column with random boundary conditions. It is assumed that the constitutive relationship between the pressure head and the unsaturated hydraulic conductivity follows the Gardner model and that the pore size distribution parameter α is a random constant in the layer. The solutions are valid for the entire unsaturated soil column. The specification of random boundary conditions allows us to easily extend the solutions to problems with multiple layers, where the statistics of soil properties in each of these layers may be different. Our solutions are verified using high resolution Monte Carlo simulations. Because of the stochastic nature of flow equations, exact analytical solutions to the problem are not available and different approximation techniques may lead to different analytical solutions in terms of the degree of complexity, the accuracy of the solutions, and the order of approximations. One advantage of the solutions given in this study over those based on the Kirchhoff transformation [*Lu and Zhang*, 2004; *Tartakovsky et al.*, 2004] is that the pressure head variance is given explicitly as a function of input variabilities rather than a function of (cross-) covariances of the intermediate Kirchhoff-transformed variables. Such explicit expressions provide more physically meaningful insights. In addition, we also investigated the effect of random boundary conditions on the flow statistics and the possible errors introduced owing to the treatment of a correlated α field as a random constant in the analytical solutions.

2. Mathematical Formulation

[5] We start from the equation for steady state flow in a one-dimensional unsaturated single-layered heterogeneous soil column, as studied by *Lu and Zhang* [2004]:

$$\frac{d}{dz}\left[K(z,\psi)\left(\frac{d\psi}{dz}+1\right)\right] = 0, \quad a \le z \le b, \tag{1}$$

with a constant pressure head at the lower boundary $z = a$,

$$\psi(a) = \Psi_a, \tag{2}$$

and a constant flux boundary at the upper boundary $z = b$,

$$K(z,\psi)\left(\frac{d\psi}{dz}+1\right)\bigg|_{z=b} = -q, \tag{3}$$

where ψ is the pressure head, Ψ_a is the specified pressure head at the bottom of the layer, $K(z, \psi)$ is the unsaturated hydraulic conductivity that depends on the pressure head, q is the flux specified at the top of the layer, and z is the vertical coordinate pointing upwards. Using this coordinate system, the flux q is negative for infiltration and positive for evaporation. Here we assume that both Ψ_a and q are specified with some uncertainties, i.e., $\Psi_a = \langle\Psi_a\rangle + \Psi'$ and $q = \langle q\rangle + q'$, where $\langle\Psi_a\rangle$ and $\langle q\rangle$ are their respective means, and $\Psi_a{}'$ and q' are their fluctuations. We should emphasize that employing random boundaries in the analytical solutions to a single layered soil is the key that allows us to extend the solutions to multiple layered soil systems.

[6] Integrating (1) and using boundary condition (3) yields

$$K(z,\psi)\left(\frac{d\psi}{dz}+1\right) = -q. \tag{4}$$

To solve the above equation, it is required to specify a constitutive relationship between the pressure head and the unsaturated hydraulic conductivity. Although the *van Genuchten* [1980] constitutive model is more accurate and widely used in deterministic simulations or numerical stochastic simulations, for mathematical convenience we adopt Gardner exponential model [*Gardner*, 1958]: $K(z, \psi) = K_s(z) \exp[\alpha\psi(z))]$, where $K_s(z)$ is the saturated hydraulic conductivity and α is the pore size distribution parameter. We further assume that K_s in the layer is a statistically homogeneous random field, and α is a random constant. The assumption of a random constant α is justified if the ratio of the correlation length of α to the thickness of the layer is relatively large [*Tartakovsky et al.*, 2003; *Lu and Zhang*, 2004]. In the limit that this ratio goes to infinity, the random constant treatment becomes exact.

[7] Equation (4) is nonlinear. In the work of *Lu and Zhang* [2004], this equation was linearized using the Kirchhoff transformation $\Phi(z) = \frac{1}{\alpha} \exp[\alpha h(z)]$, where $h = \psi + z$ is the total head, and the equation becomes $d\Phi/dz = -(q/K_s(z)) \exp(\alpha z)$. By applying perturbation analysis, they solved for moments of Φ and its associated cross-covariance in terms of statistics of soil properties. They then derived moments of the pressure head in terms of the moments of Φ and its related cross-covariance from the relationship $\alpha h(z) = \ln[\alpha\Phi(z)]$, again, using the perturbation expansion of this relationship. By substituting the expressions of Φ moments, it may be possible to express the moments of the pressure head in terms of the statistics of soil properties, but these expressions will become very lengthy and complicated. For the purpose of comparison, some of key equations by *Lu and Zhang* [2004] are listed in Appendix A.

[8] In this study, we try to derive the moments of the pressure head directly without resort to any transformations.

We formally decompose each random function as a summation of a mean and a fluctuating part: $f(z) = \ln[K_s(z)] = \langle f \rangle + f'(z)$ and $\beta = \ln[\alpha] = \langle \beta \rangle + \beta'$. Because the variability of the pressure head ψ depends on input variabilities, i.e., those of the soil properties (K_s and α) and those of the boundary conditions (Ψ_a and q), one may express ψ as an infinite series in the following form: $\psi(z) = \psi^{(0)} + \psi^{(1)} + \psi^{(2)}) + \ldots$, where the order of each term in the series is with respect to σ, which is a combination of standard derivations of the input variables. The log unsaturated hydraulic conductivity then can be written as

$$\begin{aligned} Y(z) &= \ln[K(z)] = \langle f \rangle + f'(z) + \alpha\psi(z) \\ &= Y^{(0)}(z) + Y^{(1)}(z) + \cdots, \end{aligned} \tag{5}$$

where

$$Y^{(0)}(z) = \langle f \rangle + \alpha_g \psi^{(0)}(z), \tag{6}$$

and

$$Y^{(1)}(z) = f'(z) + \alpha_g \psi^{(1)}(z) + \alpha_g \psi^{(0)}(z)\beta', \tag{7}$$

where $\alpha_g = \exp(\langle \beta \rangle)$ is the geometric mean of α. By substituting (6)-(7) and the decompositions of Ψ_a and q into (4) and (2), and separating terms at different orders up to first order, we have the zeroth-order equation

$$K_m(z,\psi)\left(\frac{d\psi^{(0)}(z)}{dz} + 1\right) = -\langle q \rangle, \tag{8}$$

subject to a boundary condition

$$\psi^{(0)}(a) = \langle \Psi_a \rangle, \tag{9}$$

where $K_m(z) = K_g \exp[\alpha_g \psi^{(0)}(z)]$ is the zeroth-order unsaturated hydraulic conductivity, and K_g is the geometric mean of the saturated hydraulic conductivity. The first-order equation is

$$K_m(z,\psi)\left[\frac{d\psi^{(1)}(z)}{dz} + Y^{(1)}(z)\left(\frac{d\psi^{(0)}(z)}{dz} + 1\right)\right] = -q', \tag{10}$$

subject to a boundary condition

$$\psi^{(1)}(a) = \Psi'_a. \tag{11}$$

2.1. Zeroth-Order Mean Pressure Head

[9] By recalling the definition of $K_m(z)$, we can rewrite (8) as

$$\frac{dK_m(z)}{dz} + \alpha_g K_m(z) = -\alpha_g \langle q \rangle, \tag{12}$$

subject to a boundary condition $K_m(a) = K_g \exp[\alpha_g \langle \Psi_a \rangle]$. This equation can be solved directly and the solution is

$$K_m(z) = K_g e^{\alpha_g(a + \langle \Psi_a \rangle - z)} - \langle q \rangle \left(1 - e^{\alpha_g(a-z)}\right). \tag{13}$$

The zeroth-order pressure head $\psi^{(0)}$ can be simply derived from (13):

$$\psi^{(0)}(z) = \frac{1}{\alpha_g} \ln\left[e^{\alpha_g(a + \langle \Psi_a \rangle - z)} - \frac{\langle q \rangle}{K_g}\left(1 - e^{\alpha_g(a-z)}\right)\right]. \tag{14}$$

Note that the zeroth-order mean head is simply the solution of the original flow equations (1)–(3) upon replacing the random variables by their mean quantities and is the same as that presented by *Lu and Zhang* [2004].

[10] Similar to the technique presented by *Yeh* [1989], the solution of the mean head for a multiple layered soil column is straightforward. The zeroth-order mean pressure head is solved sequentially from the bottom layer to the top layer. The mean head value computed at the top of a layer is taken as a constant head boundary at the bottom of the overlying layer.

2.2. Variance of Pressure Head

[11] Substituting (8) into (10) yields

$$K_m(z)\frac{d\psi^{(1)}(z)}{dz} - \langle q \rangle Y^{(1)}(z) = -q'. \tag{15}$$

By recalling (7), (15) becomes

$$\frac{d\psi^{(1)}(z)}{dz} - \frac{\alpha_g \langle q \rangle}{K_m(z)}\psi^{(1)}(z) = \frac{-q' + \langle q \rangle f'(z) + \alpha_g \langle q \rangle \psi^{(0)}(z)\beta'}{K_m(z)}. \tag{16}$$

The solution of the above first-order ordinary differential equation with boundary condition (11) is

$$\begin{aligned} \psi^{(1)}(z) = \frac{e^{\alpha_g(a-z)}}{K_m(z)} \Bigg[& K_g e^{\alpha_g \langle \Psi_a \rangle} \Psi'_a \\ & + \int_a^z \left[-q' + \langle q \rangle f'(z) + \alpha_g \langle q \rangle \psi^{(0)}(z)\beta'\right] e^{-\alpha_g(a-z)} dz \Bigg], \end{aligned} \tag{17}$$

which can be used to formulate the pressure head variance and cross-covariance between the pressure head and other variables. It is important to note that the first-order term $\psi^{(1)}$ at elevation z depends on the accumulative contribution of variation of soil properties in the interval $[a, z]$ but is independent of the soil properties above elevation z.

2.2.1. Single-Layer Soil Column

[12] For an unsaturated soil system with a single layer, we may assume that the boundary conditions q and Ψ_a are independent of medium properties f and α. The latter assumption, the independence of Ψ_a on soil properties of the layer, will be justified in section 2.2.2. If we further assume that f and α are uncorrelated and α is a random constant, then up to second order, the pressure head covariance $C\psi(y, z)$ can be derived from (17) as

$$\begin{aligned} C_\psi(y,z) &= \langle \psi^{(1)}(y)\psi^{(1)}(z) \rangle \\ &= \frac{e^{\alpha_g(2a-y-z)}}{K_m(y)K_m(z)} \Bigg[K_g^2 e^{2\alpha_g \langle \Psi_a \rangle} \sigma_{\Psi_a}^2 + \int_a^y \int_a^z \left[\sigma_q^2 + \langle q \rangle^2 C_f(y,z) \right. \\ &\quad \left. + \alpha_g^2 \langle q \rangle^2 \psi^{(0)}(y)\psi^{(0)}(z)\sigma_\beta^2\right] e^{-\alpha_g(2a-y-z)} dydz \Bigg], \end{aligned} \tag{18}$$

where the first term is the contribution of the uncertain boundary condition at the lower boundary to the head covariance at elevations z and y. We should mention that, although β' in both (16) and (17) can be z-dependent, treating β' as a function of z in (17) will result in an integral in (18) that has to be solved numerically rather than analytically.

[13] Because of symmetry of $C_\psi(y, z)$ with respect to its arguments y and z, we only need to find the solution for $y \geq z$. Integration of the first term in (18) under the double integral is trivial. For an exponential covariance function $C_f(y, z) = \sigma_f^2 \exp(-|y\text{-}z|/\lambda)$, where λ is the correlation length of f, the integration of the second and third terms can be done analytically and (18) becomes

$$\begin{aligned} C_\psi(y,z) = \frac{e^{\alpha_g(2a-y-z)}}{K_m(y)K_m(z)} \Bigg\{ & K_g^2\, e^{2\alpha_g\langle\Psi_a\rangle}\sigma_{\Psi_a}^2 + \frac{\sigma_q^2}{\alpha_g^2}\left(e^{-\alpha_g(a-y)} - 1\right) \\ & \cdot \left(e^{-\alpha_g(a-z)} - 1\right) + \frac{\langle q\rangle^2\sigma_f^2\lambda^2}{\alpha_g^2\lambda^2 - 1}\Bigg[\Bigg(1 - e^{\left(\alpha_g - \frac{1}{\lambda}\right)(z-a)} \\ & - e^{\left(\alpha_g - \frac{1}{\lambda}\right)(y-a)} + e^{\left(\alpha_g + \frac{1}{\lambda}\right)(z-a) + \left(\alpha_g - \frac{1}{\lambda}\right)(y-a)}\Bigg) \\ & - \frac{e^{2\alpha_g(z-a)} - 1}{\alpha_g\lambda}\Bigg] + K_g^2\sigma_\beta^2 F(y)F(z)\Bigg\} \end{aligned} \quad (19)$$

where

$$\begin{aligned} F(z) = {} & \psi^{(0)}(z)e^{\alpha_g\left[z+\psi^{(0)}(z)-a\right]} \\ & + \left(e^{\alpha_g\langle\Psi_a\rangle} + \frac{\langle q\rangle}{K_g}\right)(z-a) - \langle\Psi_a\rangle e^{\alpha_g\langle\Psi_a\rangle}. \end{aligned} \quad (20)$$

Equation (19) leads to the expression for the pressure head variance

$$\begin{aligned} \sigma_\psi^2(z) = {} & \sigma_{\Psi_a}^2 e^{2\alpha_g\left[H_a - \psi^{(0)}(z) - z\right]} + \frac{\sigma_q^2}{\alpha_g^2 K_m^2(z)}\left[e^{\alpha_g(a-z)} - 1\right]^2 \\ & + \frac{\langle q\rangle^2\sigma_f^2\lambda^2}{K_m^2(z)\left(\alpha_g^2\lambda^2 - 1\right)}\Bigg[1 - 2e^{\left(\alpha_g + \frac{1}{\lambda}\right)(a-z)} + e^{2\alpha_g(a-z)} \\ & - \frac{1 - e^{2\alpha_g(a-z)}}{\alpha_g\lambda}\Bigg] + \sigma_\beta^2\Bigg[\psi^{(0)}(z) + \frac{\langle q\rangle}{K_g}e^{\alpha_g\left[a-\psi^{(0)}(z)-z\right]} \\ & \cdot (z-a) - (H_a - z)e^{\alpha_g\left[H_a - \psi^{(0)}(z) - z\right]}\Bigg]^2, \end{aligned} \quad (21)$$

where $H_a = a + \langle\Psi_a\rangle$ is the total head at the lower boundary $z = a$. The first term in the right side of (21) is the contribution of uncertainty due to the variability of the constant head specified at the lower boundary to the head variance at elevation z. As z increases, this contribution decreases, as expected. The second term in the right-hand side represents the effect of uncertainty on the specified flux at the upper boundary. This term has a minimum at the low boundary, increases with elevation z, and reaches its maximum at the top boundary. The last term in (21) is the contribution of variability of α to the head variance. The third term is the contribution of the variability of the saturated hydraulic conductivity to the head variance. In case of $\alpha_g \equiv 1/\lambda$, the denominator of this term is zero and this term, denoted as $\sigma_{\psi,f}^2$, can be re-derived by taking its limit as $\alpha_g \to 1/\lambda$:

$$\sigma_{\psi,f}^2 = \frac{\langle q\rangle^2\lambda^2\sigma_f^2}{2K_m^2(z)}\left\{1 + \left[2\alpha_g(a-z) - 1\right]e^{2\alpha_g(a-z)}\right\}. \quad (22)$$

[14] Sometimes, we may be interested only in the behavior of predictive uncertainty of the pressure head within gravity-dominated regions of the unsaturated zone. For large z, (21) can be approximated by letting $z \to \infty$:

$$\sigma_\psi^2 = \frac{\sigma_q^2}{\alpha_g^2\langle q\rangle^2} + \frac{\lambda\sigma_f^2}{\alpha_g(1 + \alpha_g\lambda)} + \sigma_\beta^2\left[\Psi^{(0)}\right]^2, \quad (23)$$

where $\Psi^{(0)}$ is the zeroth-order mean pressure head in the gravity-dominated region of the layer. Note that the second term in (23), the contribution of f variability to the head variability, is identical to that of *Yeh et al.* [1985], which was derived for gravity-dominated unsaturated flow under deterministic boundary conditions.

[15] It should be noted that equation (21) may be used to compute the head variance for the one-dimensional saturated flow problem with a random constant head at one end ($z = a$) and a random constant flux at the other end, simply by setting $\alpha_g = 0$ and $\sigma_\beta^2 = 0$ in (21):

$$\begin{aligned} \sigma_\psi^2(z) = {} & \sigma_{\Psi_a}^2 + \frac{\sigma_q^2}{K_g^2}(z-a)^2 + \frac{2\langle q\rangle^2\sigma_f^2\lambda}{K_g^2} \\ & \cdot \left[(z-a) + \lambda\left(e^{(a-z)/\lambda} - 1\right)\right]. \end{aligned} \quad (24)$$

Here $\sigma_\psi^2(z)$ is the saturated head variance and $\sigma_{\Psi_a}^2$ is the uncertainty of the constant head boundary. It is interesting to see that for the saturated case the contribution of uncertainty of the constant head boundary to head variance, i.e., the first term in (24), is constant, while for the unsaturated case this contribution decreases away from the constant head boundary, as shown in (21). The difference between these two stems from the fact that the equation governing unsaturated flow is nonlinear (unsaturated hydraulic conductivity as a function of pressure head itself), while the equation for saturated flow is linear. In the latter, the uncertainty of the random constant head boundary to the head variability in the saturated system is additive. This can also be seen from mathematical derivations. The z-dependent coefficient in the first term of the right-hand-side of equation (21) can be traced back to the second term in the left-hand-side of (16), which in turn is due to the second term in the right-hand-side of (7), or to (5), which clearly shows the dependence of $Y = \ln K$ on pressure head ψ.

[16] At this point, it may be of interest to compare our solutions with those presented by *Lu and Zhang* [2004]. As stated previously by *Lu and Zhang* [2004], by using the Kirchhoff transformation, the original flow equation is transformed to an equation of the Kirchhoff-transformed variable Φ. By perturbation expansions, the mean and variance of Φ, as well as the required cross-covariances are presented as functions of soil statistics, as shown in (A3), (A4), (A5)-(A6). To obtain moments of the pressure head, one needs to transform Φ back to pressure head ψ, using the first-order approximation of the relationship $\alpha h(z) = \ln[\alpha\Phi(z)]$. In this study, the perturbation analysis is performed directly to the original flow equations and moments of the pressure head are given as explicit functions of soil statistics. Theoretically the first-order solutions from both approaches should be the same, although we are not able to verify this mathematically.

[17] In the Kirchhoff transformation approach, since the moments of the pressure head are represented in terms of moments of Φ, some errors may be introduced. *Tartakovsky et al.* [1999] compared the variance of the Kirchhoff-transformed variable derived from their analytical solution and from numerical Monte Carlo simulations, and found that the comparison was acceptable only for small variability σ_f^2 [*Tartakovsky et al.*, 1999, page 737]. However, they found that the comparison for the head variance is much better, although the head variance was computed from the moments of Φ.

[18] One advantage of the solutions from this study over those by *Lu and Zhang* [2004] is that they are more elegant and concise. In addition, since the Kirchhoff-transformed variable $\Phi = \frac{1}{\alpha}\exp(\alpha h(z))$, where h is the total head (which is always positive), the value of Φ could be very large or even cause the numerical code to crash when the geometric mean of α is very large and the layer is very thick. However, for all cases we examined, if the values of the cross-covariances of Φ and soil properties do not lead to a crash, the solutions from these two techniques are virtually the same. One advantage of the previous solutions is that the mean head solution has second-order accuracy.

2.2.2. Multilayer Soil Column

[19] For a one-dimensional soil column with n layers defined by $z_1 < z_2 < \cdots < z_{n+1}$ and given boundary conditions of an infiltration rate q at the top $z = z_{n+1}$ and a constant pressure head Ψ_{z_1} at the bottom $z = z_1$, again, solutions can be derived upward sequentially from the bottom to the top layer. An important observation is that, for one-dimensional flow problems with the boundary conditions given in this study, the head moments in any individual layer are independent of the soil properties of all overlying layers. Because the flux at the top of the bottom layer is the same as that in the top of the soil column, it is obvious that the head moments for the bottom layer can be solved alone without considering soil properties of all overlying layers. As a result, the head variability at the top of the bottom layer is uncorrelated with soil properties of the second layer, i.e., $\langle\beta'\Psi'_2\rangle = \langle\beta'\psi'(z_2)\rangle = 0$ and $\langle f'(z)\Psi'_2\rangle = \langle f'(z)\ \psi'(z_2)\rangle = 0$ for $z_2 \le z \le z_3$, where Ψ'_2 is the variability of the pressure head at the bottom of the second layer, simply because $\psi'(z_2)$ is determined from the bottom layer. This argument is valid for the rest of overlying layers. However, since the flux at the top of interface of a layer is the same as that specified at the top of the soil column, for any layer $k \ge 2$, $\langle q'\psi'(z)\rangle \ne 0$ for $z_{k-1} < z \le z_k$, i.e., $\langle q'\Psi'_k\rangle = \langle q'\psi'(z_k)\rangle \ne 0$.

[20] Based on the above reasoning, the pressure head covariance in any overlying layer k of the multiple-layer soil column can be written as

$$
\begin{aligned}
C_\psi(y,z) = \frac{e^{\alpha_g(2z_k-y-z)}}{K_m(y)K_m(z)}\Bigg[& K_g^2 e^{2\alpha_g\langle\Psi_k\rangle}\sigma^2_{\Psi_k} - K_g e^{\alpha_g\langle\Psi_k\rangle} \\
& \cdot\left[\int_a^y \langle q'\Psi'_k\rangle e^{\alpha_g(z-z_k)}dz + \int_a^z \langle q'\Psi'_k\rangle e^{\alpha_g(z-z_k)}dz\right] \\
& + \int_a^y\int_a^z \left[\sigma_q^2 + \langle q\rangle^2 C_f(y,z) + \alpha_g^2\langle q\rangle^2\psi^{(0)}(y)\psi^{(0)}(z)\sigma_\beta^2\right] \\
& \cdot e^{-\alpha_g(2z_k-y-z)}dydz\Bigg]
\end{aligned} \tag{25}
$$

where $z_k \le z < y \le z_{k+1}$, and Ψ_k is the constant head boundary at the bottom of the k^{th} layer and is determined from the underlying $(k\text{-}1)^{th}$ layer. Comparing to (18), the only difference is that the cross-covariance $\langle q'\Psi'_k\rangle$ may not be zero and should be evaluated for each sequential layer. This cross-covariance can be approximated up to first order by multiplying q' to (17), taking the mean, and carrying out the integral

$$
\langle q'\psi^{(1)}(z)\rangle = \frac{e^{\alpha_g(a-z)}}{K_m(z)}\left[\frac{\sigma_q^2}{\alpha_g}\left(1-e^{\alpha_g(z-a)}\right) + K_g e^{\alpha_g\langle\Psi_a\rangle}\langle q'\Psi'_a\rangle\right]. \tag{26}
$$

Applying this equation at the top boundary of the $(k\text{-}1)^{th}$ layer $z = z_k$ yields

$$
\begin{aligned}
\langle q'\Psi'_k\rangle = \langle q'\psi^{(1)}(z_k)\rangle = & \frac{e^{\alpha_g(z_{k-1}-z_k)}}{K_m(z_k)} \\
& \cdot\left[\frac{\sigma_q^2}{\alpha_g}\left(1-e^{\alpha_g(z_k-z_{k-1})}\right) + K_g e^{\alpha_g\langle\psi^{(0)}(z_{k-1})\rangle}\langle q'\Psi'_{k-1}\rangle\right]
\end{aligned} \tag{27}
$$

Substituting (27) into (25) and setting $y = z$ leads to the pressure head variance in the k^{th} layer:

$$
\begin{aligned}
\sigma_\psi^2(z) = & \ \sigma_\psi^2(z_k)e^{2\alpha_g(h_k-h)} + \frac{\sigma_q^2}{\alpha_g^2 K_m^2(z)}\left[e^{\alpha_g(z_k-z)}-1\right]^2 \\
& + \frac{2K_m(z_k)\langle q'\Psi'_k\rangle}{\alpha_g K_m^2(z)}\left[e^{\alpha_g(z_k-z)}-1\right]e^{\alpha_g(z_k-z)} \\
& + \frac{\langle q\rangle^2\sigma_f^2\lambda^2}{K_m^2(z)\left(\alpha_g^2\lambda^2-1\right)}\left[1-2e^{\left(\alpha_g+\frac{1}{\lambda}\right)(z_k-z)} + e^{2\alpha_g(z_k-z)}\right. \\
& \left. - \frac{1-e^{2\alpha_g(z_k-z)}}{\alpha_g\lambda}\right] + \sigma_\beta^2\left[\psi^{(0)}(z) + \frac{\langle q\rangle}{K_g}e^{\alpha_g(z_k-h)}(z-z_k)\right. \\
& \left. -(h_k-z)e^{\alpha_g(h_k-h)}\right]^2,
\end{aligned} \tag{28}
$$

where $z_k < z \le z_{k+1}$. In (28), h_k and $\sigma_\psi^2(z_k)$ are respectively the total head and the head variance specified at the bottom of the k^{th} layer, both of which are determined from the $(k\text{-}1)^{th}$ layer for $k \ge 2$. The parameters K_g, α_g, σ_f^2, λ, and σ_β^2 in (28) refer to soil properties of the k^{th} layer. The index k in these parameters have been omitted for simplicity in mathematical representation. If $\alpha_g = 1/\lambda$ for some layers, the term on the third line of (28) for these layers should be replaced by the expression in (22).

2.3. Statistics of Effective Water Content

[21] The effective water content can be written as [*Mualem*, 1976]

$$
\theta_e(z) = (\theta_s-\theta_r)\left[e^{\alpha\psi(z)/2}(1-0.5\alpha\psi(z))\right]^{2/(m+2)}, \tag{29}
$$

where θ_s and θ_r are the saturated and residual water contents, respectively, and m is a parameter related to tortuosity of the porous media, usually set to be 0.5. Here we assume that both θ_s and θ_r are deterministic variables. By writing $\theta_e = \theta_e^{(0)} + \theta_e^{(1)} + \cdots$, substituting this and the decompositions of α and ψ into (29), and separating the equation at different order, one obtains the zeroth-order approximation

$$\theta_e^{(0)}(z) = (\theta_s - \theta_r)\left[e^{\alpha_g \psi^{(0)}(z)/2}\left(1 - 0.5\ \alpha_g \psi^{(0)}(z)\right)\right]^{2/(m+2)}, \quad (30)$$

and the first-order term

$$\theta_e'(z) = -\frac{\alpha_g^2 \psi^{(0)}(z)\theta_e^{(0)}(z)}{(m+2)\left(2 - \alpha_g \psi^{(0)}(z)\right)}\left[\psi'(z) + \beta'\psi^{(0)}(z)\right], \quad (31)$$

which leads to an expression for the variance of the effective water content

$$\sigma_{\theta_e}^2 = \left[\frac{\alpha_g^2 \psi^{(0)}(z)\theta_e^{(0)}(z)}{(m+2)\left(2 - \alpha_g \psi^{(0)}(z)\right)}\right]^2 \left[\sigma_\psi^2(z) + \sigma_\beta^2\left[\psi^{(0)}(z)\right]^2 + 2\psi^{(0)}(z)C_{\beta\psi}(z)\right]. \quad (32)$$

The one-point cross-covariance $C_{\beta\psi}(z)$ can be derived by multiplying β' to (17), taking ensemble means, and carrying out integration:

$$C_{\beta\psi}(z) = \sigma_\beta^2\left[\psi^{(0)}(z) + \frac{\langle q\rangle}{K_g}e^{\alpha_g\left[a - \psi^{(0)}(z) - z\right]} \cdot (z-a) - (H_a - z)e^{\alpha_g\left[H_a - \psi^{(0)}(z) - z\right]}\right]. \quad (33)$$

2.4. Statistics of Log Hydraulic Conductivity

[22] By writing $K(z) = K^{(0)}(z) + K^{(1)}(z) + \cdots$ and recalling $K(z) = K_s \exp(\alpha\psi)$ and $\psi(z) = \psi^{(0)}(z) + \psi^{(1)}(z) + \cdots$, we have

$$K^{(0)}(z) + K^{(1)}(z) + \cdots = K_g(1 + f' + \cdots) \cdot e^{\alpha_g \psi^{(0)}(z)}\left[1 + \alpha_g \psi^{(1)}(z) + \alpha_g \psi^{(0)}(z)\beta' + \cdots\right]. \quad (34)$$

Separating (34) at different orders leads to

$$K^{(0)}(z) = K_m(z) = K_g e^{\alpha_g \psi^{(0)}(z)}, \quad (35)$$

and the first-order approximation

$$K^{(1)}(z) = K_m(z)Y^{(1)}(z) = K_m(z)\left[f' + \alpha_g \psi^{(1)}(z) + \alpha_g \psi^{(0)}(z)\beta'\right]. \quad (36)$$

The latter allows us to formulate the variance of the unsaturated hydraulic conductivity

$$\sigma_K^2(z) = K_m^2(z)\sigma_Y^2(z) = K_m^2(z)\left\{\sigma_f^2(z) + 2\alpha_g C_{f\psi}(z) + \alpha_g^2 \cdot \left[\sigma_\psi^2(z) + 2\psi^{(0)}(z)C_{\beta\psi}(z) + \sigma_\beta^2\left[\psi^{(0)}(z)\right]^2\right]\right\}, \quad (37)$$

where the one-point cross-covariance $C_{f\psi}(z)$ can be derived by multiplying $f'(z)$ to (17), taking ensemble means, and carrying out integration:

$$C_{f\psi}(z) = \frac{\langle q\rangle \lambda \sigma_f^2}{(\alpha_g\lambda + 1)K_m(z)}\left[1 - e^{\left(\alpha_g + 1/\lambda\right)(a-z)}\right]. \quad (38)$$

3. Illustrative Examples

[23] In this section, we demonstrate the accuracy of our first-order analytical solutions of the mean pressure head and the head variance for one-dimensional steady state unsaturated flow in a hypothetical layered soil column (base case), by comparing our results with those from Monte Carlo simulations. As mentioned previously, the solutions from this paper and those by *Lu and Zhang* [2004] are virtually identical, if parameter values will not lead to a crach for the previous solutions. For parameter values in the following base case, it has been shown that the solutions from two techniques are the same and the accuracy of the previous solutions has been demonstrated by comparing with Monte Carlo simulations, as shown by *Lu and Zhang* [2004]. As a result, we only demonstrate the comparison of the water content statistics that have not been compared by *Lu and Zhang* [2004].

[24] The effect of uncertain boundary conditions (variable constant head at the bottom and variable flux at the top boundary) on the variance of head has been investigated in this section.

3.1. Base Case

[25] Similar to the base case by *Lu and Zhang* [2004], we consider a one-dimensional heterogeneous soil column with three layers. The length of the soil column is 20 m and the thickness of these layers (from the bottom to the top layer) is 10 m, 5 m, and 5 m, respectively. The column is uniformly discretized into 400 line segments (one-dimensional elements) of 0.05 m in length. The origin of the vertical coordinate is set at the bottom of the column. The mean total head is prescribed at the bottom as $\langle H_a\rangle$ = 0.0 m (i.e., $\langle\Psi_a\rangle$ = 0.0, water table) and $\sigma_{H_a}^2 = \sigma_{\Psi_a}^2 \equiv 0$, and the mean infiltration rate at the top boundary is given as $\langle q\rangle$ = -0.002 m/day with a standard deviation of σ_q = 0.0004 m/day, i.e., coefficient of variation CV_q = 20%. Here the negative mean flux $\langle q\rangle$ represents infiltration. In this base case, we choose a relatively small variability of the infiltration rate to ensure that the Monte Carlo simulations, which are conducted to validate the first-order analytical solutions, will converge. The means of the log saturated hydraulic conductivity for three layers are given as $\langle f\rangle$ = 0.0, 1.0, and 0.0, respectively, with CV_{Ks} = 100% (σ_f^2 = 0.693) for all layers. The correlation length of the log hydraulic conductivity is λ = 1.0 m for all layers. Unless mentioned explicitly, the logarithm of the pore size distribution parameter for three layers are taken as random constants and their statistics are given as $\langle\beta\rangle$ = 0.693, 1.099, and 0.405, respectively, which gives the geometric mean of α_g = 2.0 m^{-1}, 3.0 m^{-1}, and 1.5 m^{-1}. These α values are in the range of gravelly sandy soils [*Khaleel and Relyea*, 2001]. The variability of α is given as CV_α = 10% for all layers. The saturated and residual water contents are considered as deterministic variables and are taken as θ_s = 0.3 and θ_r = 0.02 for all layers.

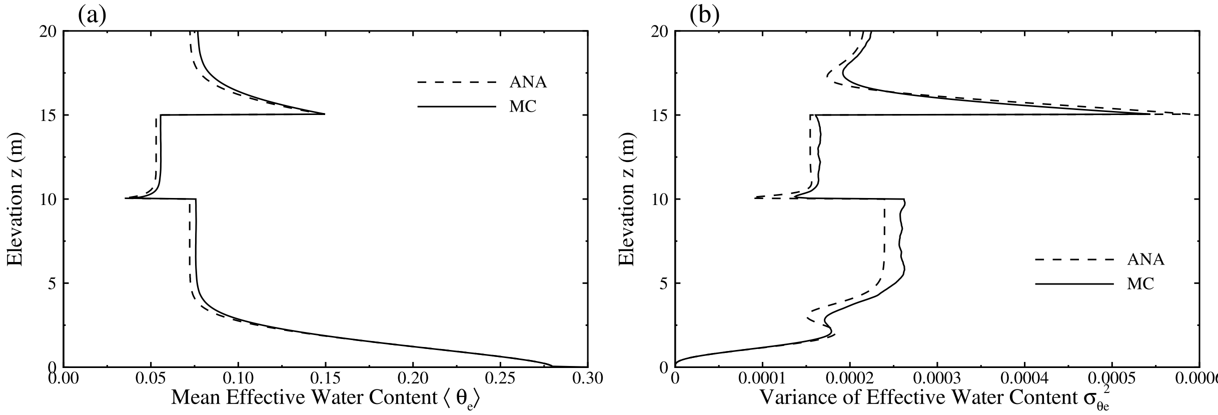

Figure 1. Comparisons of analytical solutions and Monte Carlo simulation results: (a) mean, and (b) variance of the effective water content for the base case.

[26] To evaluate the accuracy of the first-order analytical solutions, we conduct Monte Carlo simulations for comparison purposes. For three layers, we generate three sets of realizations, each of which includes 10,000 one-dimensional unconditional realizations, using the sequential Gaussian random field generator *sgsim* from GSLIB [*Deutsch and Journel*, 1998]. Each set of these realizations has been tested separately by comparing their sample statistics (the mean, variance, and correlation length) against the specified mean and covariance functions. The comparisons show that the generated random fields reproduce the specified mean and covariance structure well. Realizations of the log hydraulic conductivity fields for the whole column (three segments) are then composed using three realizations taken from each set. The log pore size distribution parameters for the three layers are generated from a random number generator. In the case of uncertain boundary conditions (a random infiltration rate at the upper boundary and/or a random constant head at the lower boundary), boundary values are also generated using the random number generator.

[27] The steady state unsaturated flow equation, i.e., equation (1) subject to boundary conditions (2)–(3), is solved sequentially from the bottom to the top using Yeh's algorithm [*Yeh*, 1989] for each realization of the log hydraulic conductivity field and the pore size distribution parameter for the three layers. In the case that the uncertainties in the infiltration rate and/or constant head boundary condition (at the bottom) are involved (as in other examples), randomly generated values of the infiltration rate and constant head will be used. If a solution for pressure head contains any positive values, the realization corresponding to this solution is simply removed. Since the parameter variations in the base case are relatively small, only a few out of 10,000 Monte Carlo simulation runs are removed, which does not significantly affect flow statistics. The sample statistics for the flow field, i.e., the mean prediction of head and its associated uncertainty (variance) are then computed from the rest of realizations. These statistics are considered the "true" solutions that are used to compare against the derived analytical solutions of the moment equations.

[28] The moments of the effective water content is illustrated in Figure 1, which shows that the analytical solutions can reproduce the true solutions very well for this case. It is interesting to note that in the second layer the effective water content increases from its lower boundary upwards. This is partially ascribed to the lower mean pressure head at the bottom of this layer as a direct effect of lower pressure head in the underlying layer.

3.2. Uncertain Boundary Flux

[29] To investigate the effect of boundary flux uncertainty on the mean flow field and the head variance, we conduct several numerical experiments using (28) with different magnitudes of the coefficient of variation in q, CV_q = 0%, 50%, 100%, and 200%, while the variabilities of the log hydraulic conductivity f and the log pore size distribution parameter β remain the same as in the base case. Because the variation of the infiltration rate does not affect the zeroth-order mean flow field, we are only concerned with the pressure head variance in our discussion. Figure 2 illustrates the effect of the variability of q on the head variance. It is seen from Figure 2 that after excluding the effect of the variabilities of f and α, the contribution of q variability to the pressure head variance is linearly proportional to the square of CV_q, i.e., linearly proportional to σ_q^2. This can also be seen from equation (28).

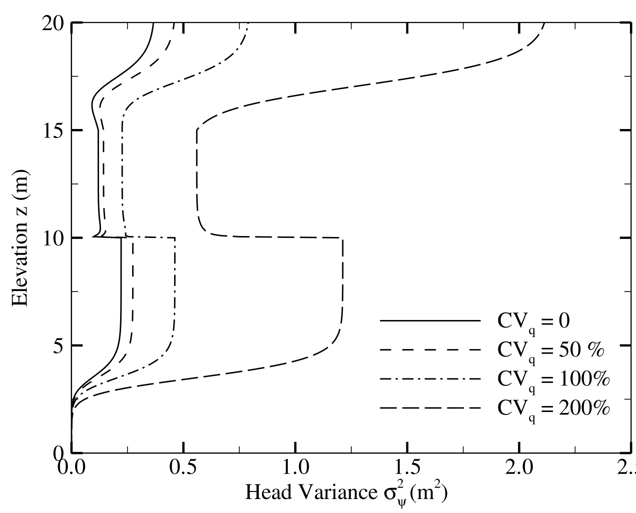

Figure 2. The effect of the variability of the infiltration rate q on the pressure head variance.

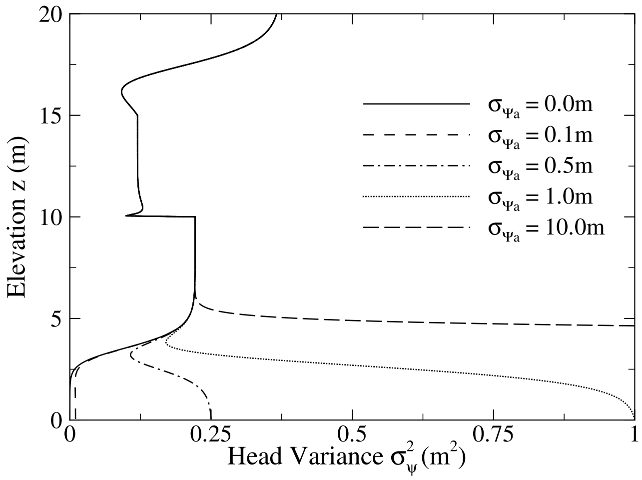

Figure 3. The effect of the uncertainty of the specified constant head at the lower boundary on the pressure head variance.

3.3. Uncertain Constant Head Boundary

[30] In most practical problems, it is not easy to precisely specify the pressure head at the lower boundary of an unsaturated soil column. Or sometimes, we are not able to specify the exact location of the water table. As a result, a constant head at the lower boundary should be specified with some uncertainty. We are interested in how this uncertainty will affect our prediction of the mean head and its associated uncertainty. Figure 3 shows the profile of the pressure head variance, as computed from (28), for different magnitudes of uncertainty on the prescribed constant head at the lower boundary: σ_{ψ_a} = 0.0 m, 0.1 m, 0.5 m, 1.0 m, and 10.0 m. An important observation from the figure is that the contribution of the boundary head uncertainty to the pressure head variance decreases with elevation z and this contribution reduces to zero in the gravity-dominated region. The implication from this observation is that, once the flow in the upper portion of a layer reaches the gravity-dominated regime, the uncertainty of the prescribed head at the bottom of the column will not have any effect on the pressure head uncertainty in all overlying layers.

[31] Another way to look at the effect of uncertainty in the constant head boundary at the bottom of the column on the predictive head variance is to specify different values of the head boundary and to see how the changes to the prescribed head will affect the head uncertainty in the column. Figure 4 shows profiles of the mean head (Figure 4a) and the head variance (Figure 4b) for different values of Ψ_a. It is seen from the figure that the variation of the constant head specified at the bottom boundary does have an effect on the predictions. However, if the flow in the upper portion of a layer has reached the gravity-dominated regime, the variation in the constant head value does not have any effect on the overlying layers.

3.4. Relative Contribution of Variabilities in K_s, α, and q

[32] We also conducted three numerical simulations to investigate the relative contribution of the variability of f, β, and q to the pressure head variance. In each simulation, we only allow variation in one of these three parameters with a coefficient of variation CV_f = 50.0%, CV_α = 15%, and CV_q = 50%, while all other parameters are the same as in the base case. The results are illustrated in Figure 5, where the dashed curve, dash-dotted curve, and dotted curve represent the pressure head variance due to the variability of α, K_s, and q, respectively. The solid curve in Figure 5 stands for the pressure head variance due to the variabilities of all three parameters.

[33] It is seen that under the condition of mutually independent K_s, α, and q, the contribution of the variability in each parameter to the pressure head variance is additive, namely, the pressure head variance due to the variabilities of all three parameters equals the sum of the three pressure head variances due to the variability of the individual parameter. In addition, it seems that the variability in the pore size distribution α has the largest contribution to the pressure head variance, compared to other parameters with the same magnitude of coefficients of variation. The finding that unsaturated flow is most sensitive to the variability in α is consistent with the earlier observations made by *Zhang et al.* [1998], where only the effects of f and α were studied.

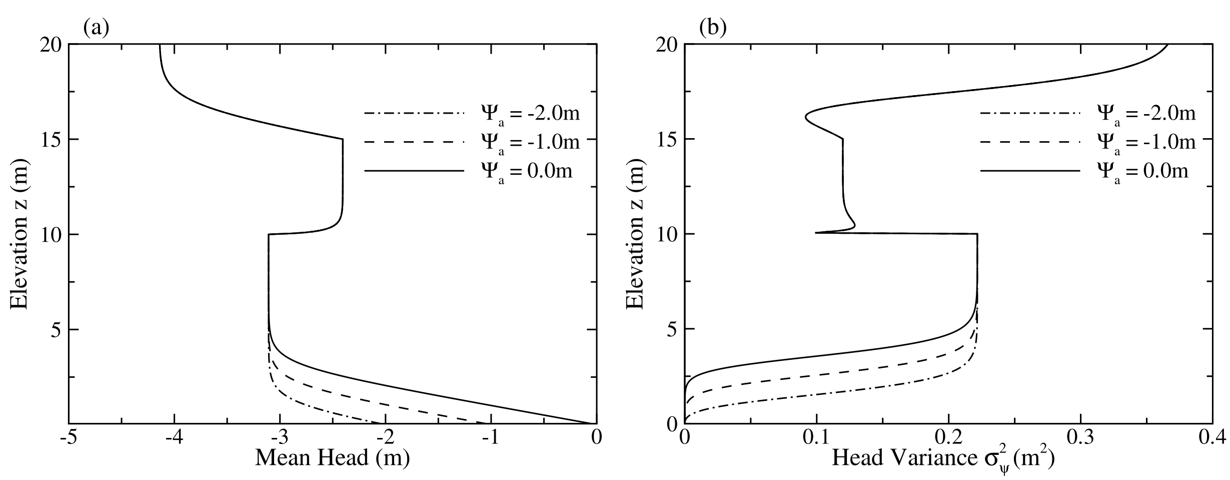

Figure 4. The effect of various values of the specified constant head at the lower boundary on the pressure head variance.

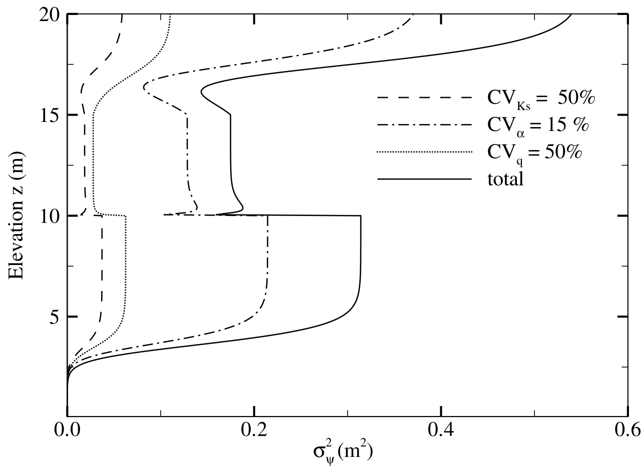

Figure 5. Relative contribution of input variabilities on the pressure head variance.

3.5. Effect of Correlation Length of α

[34] To investigate the possible errors introduced due to treating a correlated α field as a random constant in the analytical solutions, we conduct two more sets of Monte Carlo simulations. The layer configuration and parameter values used are the same as in the base case except for that the α field here is a correlated random spatial function rather than a random constant. In the first case, the correlation length is set to λ_β = 1.0 m for all three layers and in the second case λ_β = 5.0 m. Realizations of β are generated similarly as described for generation of ln K_s fields. The results are illustrated in Figure 6, where $\lambda_\beta = \infty$ represents Monte Carlo simulations with a random constant β. Figure 6a clearly shows that treating a correlated α field as a random constant in Monte Carlo simulations does not significantly affect the mean pressure head profile and that the mean pressure head from the analytical solution matches these Monte Carlo results very well.

[35] Figure 6b compares the head variance derived from Monte Carlo simulations of various correlation lengths of α and from the analytical solutions in which α is treated as a random constant. It is seen from the figure that in the case of a small λ_β, treating the correlated α field as a random constant will introduce some noticeable error (comparing the dashed line and solid line in Figure 6b). However, if the correlation length λ_β is large, treating α as a random constant in the analytical solutions is a reasonable approximation.

4. Summary and Conclusions

[36] In this study, we directly solved the same problem as studied by *Lu and Zhang* [2004] without employing any transformations. More specifically, we derived first-order analytical solutions to the mean pressure head and the head variability as well as the moments of the effective water content and the unsaturated hydraulic conductivity for one-dimensional steady state unsaturated flow in a layered, randomly heterogeneous soil column under random boundary conditions (a prescribed constant head at the bottom and a flux at the top boundary). It is assumed in the solutions that the constitutive relation between the unsaturated hydraulic conductivity and the pressure head, and between the effective water content and the pressure head follow Gardner-Russo model. The solutions are not limited to the gravity-dominated flow regime but are valid for the entire unsaturated zone. The accuracy of these solutions is verified using Monte Carlo simulations. Numerical examples show that these solutions are valid for relatively large variabilities in soil properties.

[37] In practice, it is hard to specify precisely the constant pressure head and its associated uncertainty at the lower boundary. An important observation from this study is that once the flow reaches a gravity-dominated regime in a layer, the actual value of the pressure head and its variability at the lower boundary do not have any effects on the pressure head statistics in all overlying layers.

[38] The analytical solution confirms our previous conclusion that the variability of the pore size distribution parameter α makes a more important contribution to the head variability than the variabilities of the log hydraulic conductivity and the infiltration rate.

[39] Monte Carlo simulations with a spatially correlated pore size distribution parameter α of various correlation lengths indicate that treating α as a random constant is a reasonable approximation if the correlation length of α is relatively large.

[40] One of the advantages of the solution presented in this study over the previous one [*Lu and Zhang*, 2004] is that the head variance is explicitly expressed as a function of input variabilities, i.e, those of the log hydraulic conductivity, the pore size distribution parameter, and

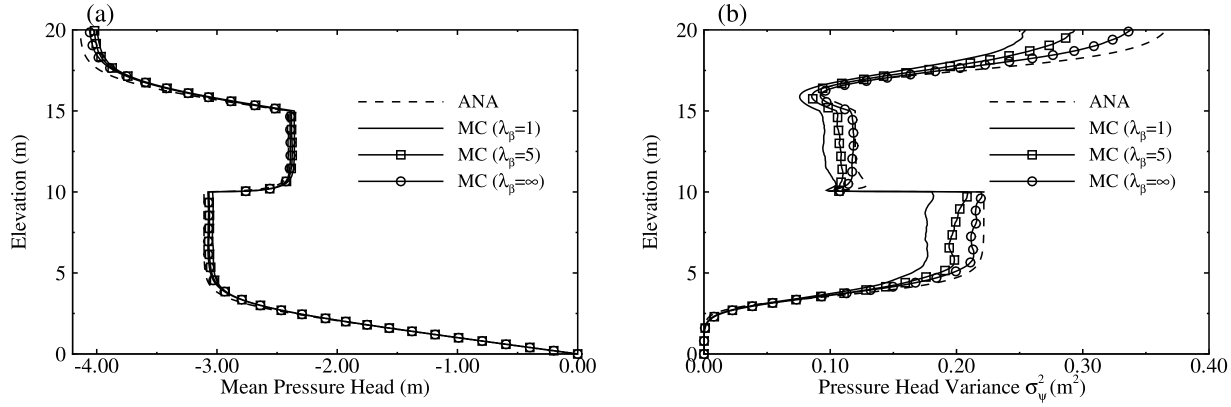

Figure 6. Comparisons of (a) mean and (b) head variance derived from analytical solutions and those from Monte Carlo simulations with various of correlation lengths of the α field.

boundary conditions. In addition, when α_g is large, the solution by *Lu and Zhang* [2004] may cause numerical instability or even numerical overflow, because in that solution the moments of the intermediate, Kirchhoff-transformed variable Φ is a function of $\exp(\alpha_g z)$. One limitation of the new solution is that the mean pressure head is only approximated to zeroth order, while in the solution of *Lu and Zhang* [2004] the mean head is accurate to the second order.

Appendix A

[41] For the purpose of comparison, we provide some key results from *Lu and Zhang* [2004]. The mean total head is approximated to second order in terms of soil variabilities, $\langle h(z)\rangle \approx h^{(0)}(z) + \langle h^{(2)}(z)\rangle$,

$$h^{(0)}(z) = \frac{1}{\alpha_g}\ln\left[\alpha_g\Phi^{(0)}(z)\right] = \frac{1}{\alpha_g}\ln\left[e^{\alpha_g\langle H_a\rangle} - \frac{\langle q\rangle}{K_g}\left(e^{\alpha_g z} - e^{\alpha_g a}\right)\right], \tag{A1}$$

and

$$\langle h^{(2)}(z)\rangle = -\frac{\sigma_\beta^2}{\alpha_g} + \frac{1}{2}\sigma_\beta^2 h^{(0)}(z) - \frac{\langle\beta'\Phi^{(1)}(z)\rangle}{\alpha_g\langle\Phi^{(0)}(z)\rangle} + \frac{\langle\Phi^{(2)}(z)\rangle}{\alpha_g\langle\Phi^{(0)}(z)\rangle} - \frac{\sigma_\Phi^2(z)}{2\alpha_g\langle\Phi^{(0)}(z)\rangle^2}, \tag{A2}$$

where $\langle H_a\rangle = \langle\psi_a\rangle + a$ is the total head at the lower boundary, and other terms are defined as

$$\langle\Phi^{(0)}(z)\rangle = \Phi_a - \frac{\langle q\rangle}{\alpha_g K_g}\left(e^{\alpha_g z} - e^{\alpha_g a}\right), \tag{A3}$$

$$\langle\Phi^{(2)}(z)\rangle = \frac{\Phi_a}{2}\alpha_g^2\sigma_{H_a}^2 + \frac{\Phi_a}{2}\left(1 - \alpha_g\langle H_a\rangle + \alpha_g^2\langle H_a\rangle^2\right)\sigma_\beta^2 - \frac{\langle q\rangle\sigma_Y^2}{2\alpha_g K_g}\left(e^{\alpha_g z} - e^{\alpha_g a}\right) - \frac{\langle q\rangle\sigma_\beta^2}{2\alpha_g K_g}\left[\left(1 - \alpha_g z + \alpha_g^2 z^2\right)\cdot e^{\alpha_g z} - \left(1 - \alpha_g a + \alpha_g^2 a^2\right)e^{\alpha_g a}\right]. \tag{A4}$$

$$\begin{aligned}\sigma_\Phi^2(z) = {}& \Phi_a^2\alpha_g^2\sigma_{H_a}^2 + \Phi_a^2\left(\alpha_g\langle H_a\rangle - 1\right)^2\sigma_\beta^2 - \frac{2\Phi_a}{K_g}\langle q'H_a'\rangle\left[e^{\alpha_g z} - e^{\alpha_g a}\right] \\ & - \frac{2\langle q\rangle\Phi_a\sigma_\beta^2}{\alpha_g K_g}\left(\alpha_g\langle H_a\rangle - 1\right)\left[\left(\alpha_g z - 1\right)e^{\alpha_g z} - \left(\alpha_g a - 1\right)e^{\alpha_g a}\right] + \frac{\langle q\rangle^2\sigma_\beta^2}{\alpha_g^2 K_g^2}\left[\left(\alpha_g z - 1\right)e^{\alpha_g z} - \left(\alpha_g a - 1\right)e^{\alpha_g a}\right]^2 + \frac{\sigma_q^2}{\alpha_g^2 K_g^2}\left(e^{\alpha_g z} - e^{\alpha_g a}\right)^2 \\ & + \frac{\langle q\rangle^2\lambda\sigma_Y^2 e^{2\alpha_g a}}{\alpha_g K_g^2\left(1 - \alpha_g^2\lambda^2\right)}\left[2\alpha_g\lambda e^{\left(\alpha_g - 1/\lambda\right)(z-a)} - \left(1 - \alpha_g\lambda\right)e^{2\alpha_g(z-a)} - \left(1 + \alpha_g\lambda\right)\right]\end{aligned} \tag{A5}$$

$$\langle\beta'\Phi^{(1)}(z)\rangle = \Phi_a\left(\alpha_g\langle H_a\rangle - 1\right)\sigma_\beta^2 - \frac{\langle q\rangle\sigma_\beta^2}{\alpha_g K_g}\left[\left(\alpha_g z - 1\right)e^{\alpha_g z} - \left(\alpha_g a - 1\right)e^{\alpha_g a}\right], \tag{A6}$$

and $\Phi_a = \frac{1}{\alpha_g}\exp(\alpha_g\langle H_a\rangle)$. The variance of the pressure head reads as

$$\sigma_\psi^2(z) = \sigma_h^2(z) = \frac{\sigma_\beta^2}{\alpha_g^2}\left(1 - \alpha_g h^{(0)}(z)\right)^2 + \frac{2\left(1 - \alpha_g h^{(0)}(z)\right)}{\alpha_g^2\langle\Phi^{(0)}(z)\rangle}\langle\beta'\Phi^{(1)}(z)\rangle + \frac{\sigma_\Phi^2(z)}{\alpha_g^2\langle\Phi^{(0)}(z)\rangle^2}. \tag{A7}$$

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